# NONLINEAR MODELS OF DEFORMABLE MEDIA WITH COUPLE-STRESSES

L. I. Shkutin

UDC 539.30

In the classical deformable-continuum model developed by Cauchy the strain is completely determined by the vector field of displacements, and the stress state by the tensor field of stresses. Although real media have a discrete structure, the classical model very successfully describes the stress and strain distribution in sufficiently smooth domains under sufficiently steady loading. However, this model suffers a loss of precision when the conditions of smoothness of the domains break down and the load gradients increase. The discreteness of the structure of real media becomes significant in such situations. Consistency with experiment can be regained at the cost of a departure from the classical continuum model and its replacement with other models that comply more nearly with experiment. Clearly, the most universal model is the model of a medium as a discrete system of particles bound by definite interaction forces. A simpler approach is to modify the classical model in such a way as to preserve the hypothesis of continuity of the medium while imparting to it smooth special properties of a discrete-structured medium. The most highly developed model of a deformable medium to date is the model originally formulated by the brothers Cosserat, namely a medium subjected to couple stresses. The Cosserat continuum is a medium whose strain is determined by kinematically independent vector fields of displacements and rotations and whose stress state is determined by tensor fields of internal forces and couples. The most complete formulation of the equations corresponding to this model is clearly attributable to Toupin [1].

In this article we propose a modified version of the nonlinear equations formulated in [1] for a thermomechanical couple-stress continuum. The modification is made by introducing a vector field of finite rotations, which greatly simplifies the nonlinear formulation of the kinematic equations and brings it ultimately close to the linear formulation. For the formulated system of nonlinear equations we indicate kinematic and dynamic relations that result in simpler nonlinear models of deformable media.

A domain of three-dimensional Euclidian space occupied by a continuum at the initial time is parametrized by the Lagrangian coordinates $t_N$ (upper-case Latin subscripts take the values 1, 2, 3).

Let $(t) \equiv (t_1, t_2, t_3)$ be an arbitrary material point of the continuum, a(t) its initial radius vector relative to a fixed (reference) point in space, $\nabla_N$ the partial differentiation operator with respect to the variable $t_N$, $\mathbf{A}_{(N)}(t) \equiv \nabla_N \mathbf{a}$ an initial coordinate basis defined at point (t), $A_{(MN)}(t) \equiv \mathbf{A}_{(M)} \cdot \mathbf{A}_{(N)}$ the metric tensor of the initial basis, $D_{(LMN)}(t) \equiv \mathbf{A}_{(L)} \cdot (\mathbf{A}_{(M)} \times \times \mathbf{A}_{(N)})$ the discriminant tensor of the initial basis, and $\nabla_{(N)}$ the covariant differentiation operator with respect to the variable $t_N$ in the initial basis.

At the initial time the continuum is subjected to mechanical and (or) thermal effects, which induce strain-accompanied motion of its material points. The closed system of equations describing the motion of the continuum comprises the kinematic, dynamic, and constructive equations.

1. The Lagrangian description of the strain-accompanied motion of the continuum reduces the problem to a continuous sufficiently smooth transformation of the coordinate basis.

Let $t_0$ denote the time variable, $\mathbf{a}_0(t_0, t)$ the instantaneous radius of the material point (t) relative to the reference point in space, $\mathbf{A}_{\{N\}}(t_0, t) \equiv \nabla_N \mathbf{a}_0$ the instantaneous coordinate basis, and $\nabla_0$ the differentiation operator with respect to the parameter $t_0$.

According to the concept of the couple-stress continuum, a displacement-independent local rotation of the basis is admitted in the transformation of the initial to the instantaneous basis. Consequently, for the mathematical description of such a transformation it is appropriate to invoke the notion of Biot concerning the segregation of a rigid rotation apart from the general transformation of basis [2]. This notion has been realized in the formulation of the second-order theory of a classical (couple-free) elastic continuum.

The generalization of Biot's notion to a couple-stress continuum permits the transformation of the initial to the instantaneous basis to be represented by the superposition of two independent successive transformations: 1) a local rigid rotation, taking the initial basis into a certain (rotated) basis $\mathbf{A}_{[N]}(t_0, t)$; 2) transformation of the rotated basis into the instantaneous basis.

---

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 111-117, November-December, 1980. Original article submitted February 28, 1980.

 0021-8944/80/2106- 0826$07.50 

The local rigid rotation of the basis is characterized by the angle of rotation $\vartheta(t_0, t)$ about the instantaneous local axis. It is more practical, however, to characterize it by a local rotation vector $\mathbf{V}(t_0, t)$, which is directed along this axis and has length

$$|\mathbf{V}| = 2\left|\operatorname{tg}\left(\frac{1}{2}\vartheta\right)\right|.$$

The rigid-rotation transformation of the basis is expressed in terms of the rotation vector by the mutually inverse Rodrigues formulas [3]

$$\begin{aligned} \mathbf{A}_{[N]} &= \mathbf{A}_{(N)} + \frac{1}{f}\left(\mathbf{A}_{(N)} + \frac{1}{2}\mathbf{V} \times \mathbf{A}_{(N)}\right), \\ \mathbf{A}_{(N)} &= \mathbf{A}_{[N]} - \frac{1}{f}\left(\mathbf{A}_{[N]} - \frac{1}{2}\mathbf{V} \times \mathbf{A}_{[N]}\right), \quad f = 1 + \frac{1}{4}\mathbf{V}\cdot\mathbf{V}. \end{aligned} \tag{1.1}$$

The metric and discriminant tensors are invariant under rigid rotation, so that

$$\mathbf{A}_{[M]}\cdot\mathbf{A}_{[N]} = A_{(MN)}, \quad \mathbf{A}_{[M]} \times \mathbf{A}_{[N]} = D_{(LMN)}\mathbf{A}^{[L]}. \tag{1.2}$$

As a result, the subscript "juggling" operation in the rotated basis is realized by means of the initial metric tensor.

The solution of either of the equations (1.1) for the rotation vector yields the equation

$$\mathbf{V} = \frac{1}{2} f \mathbf{A}^{(N)} \times \mathbf{A}_{[N]},$$

which describes the rotation vector in terms of the rotated basis. A consequence is the set of equalities

$$\mathbf{V}\cdot\mathbf{A}_{[N]} = \mathbf{V}\cdot\mathbf{A}_{(N)},$$

which states that the components of the rotation vector coincide in the initial and rotated bases.

The field of finite rotations formed by the vector function $\mathbf{V}(t_0, t)$, preserving the space metric, can only impart curvatures to lines and surfaces immersed in the given space (in particular, the coordinate lines and surfaces). A measure of curvature is afforded by the vectors $\nabla_{(L)}\mathbf{A}_{[M]}$. However, there is a simpler measure, which is expressed in terms of these vectors. Thus, covariant differentiation of the scalar product (1.2) yields the equation

$$\mathbf{A}_{[N]}\cdot\nabla_{(K)}\mathbf{A}_{[M]} = -\mathbf{A}_{[M]}\cdot\nabla_{(K)}\mathbf{A}_{[N]},$$

which implies the representation

$$\nabla_{(K)}\mathbf{A}_{[M]} = \mathbf{V}_{(K)} \times \mathbf{A}_{[M]}. \tag{1.3}$$

The vectors $\mathbf{V}_{(K)}(t_0, t)$ introduced in this way define a simpler and more natural measure of curvature and are therefore logically referred to as the curvature-strain vectors of the continuum.

The inversion of Eq. (1.3) yields the equation

$$\mathbf{V}_{(K)} = \frac{1}{2}\mathbf{A}^{[M]} \times \nabla_{(K)}\mathbf{A}_{[M]}, \tag{1.4}$$

which expresses the curvature-strain vectors in terms of the vectors $\nabla_{(K)}\mathbf{A}_{[M]}$. Equations (1.1) can be used to derive from (1.4) the following expression for the curvature-strain vectors in terms of the rotation vector:

$$\mathbf{V}_{(K)} = \frac{1}{f}\left(\nabla_K\mathbf{V} + \frac{1}{2}\mathbf{V} \times \nabla_K\mathbf{V}\right). \tag{1.5}$$

The curvature-strain vectors have tensor components and form a tensor field of nonlinear curvature-strains of the continuum.

The transformation of the rotated into the instantaneous basis generates vectors $\mathbf{U}_{(K)}(t_0, t)$, which are defined by the equation [$\mathbf{U}(t_0, t)$ is the displacement vector]

$$\mathbf{U}_{(K)} \equiv \mathbf{A}_{(K)} - \mathbf{A}_{[K]} = \nabla_K\mathbf{U} + \mathbf{A}_{(K)} - \mathbf{A}_{[K]} \tag{1.6}$$

and afford a measure of the strain-induced variation of the metric tensor of the given space. These vectors are appropriately called the metric-strain vectors of the continuum.

Substituting expressions (1.1) into (1.6), we obtain the equivalent equations

$$\mathbf{U}_{(K)} = \nabla_K \mathbf{U} - \frac{1}{f} \mathbf{V} \times \left(\mathbf{A}_{(K)} + \frac{1}{2} \mathbf{V} \times \mathbf{A}_{(K)}\right) = \\ = \nabla_K \mathbf{U} - \frac{1}{f} \mathbf{V} \times \left(\mathbf{A}_{[K]} - \frac{1}{2} \mathbf{V} \times \mathbf{A}_{[K]}\right), \tag{1.7}$$

which expresses the metric-strain vectors in terms of the displacement and rotation vectors.

The metric-strain vectors have tensor components and form a tensor field of nonlinear metric-strains of the continuum.

The inverses of Eqs. (1.5) and (1.6)

$$\nabla_K \mathbf{V} = \mathbf{V}_{(K)} + \frac{1}{2} \mathbf{V}_{(K)} \times \mathbf{V} + \frac{1}{4} (\mathbf{V}_{(K)} \cdot \mathbf{V}) \mathbf{V},$$
$$\nabla_K \mathbf{U} = \mathbf{U}_{(K)} + \mathbf{A}_{[K]} - \mathbf{A}_{(K)}$$

can be regarded as the system of equations describing the rotation and displacement vectors in terms of the strain vectors (tensors). The conditions for integrability (compatibility) of this system

$$D^{(KLM)} \nabla_{(K)} \nabla_L \mathbf{V} \equiv 0, \; D^{(KLM)} \nabla_{(K)} \nabla_L \mathbf{U} \equiv 0$$

are reduced by (1.3) to the equations

$$D^{(KLM)} \left(\nabla_{(K)} \mathbf{V}_{(L)} - \frac{1}{2} \mathbf{V}_{(K)} \times \mathbf{V}_{(L)}\right) = 0, \\ D^{(KLM)} (\nabla_{(K)} \mathbf{U}_{(L)} + \mathbf{V}_{(K)} \times \mathbf{A}_{[L]}) = 0, \tag{1.8}$$

which have the significance of continuity conditions for the deformable medium.

To complete the formulation of the kinematic equations we have only to determine the rates of time variation of the kinematic vectors defined above.

Inasmuch as the motion of the rotated basis is spherical, it can be represented in terms of the angular velocity vector $\mathbf{V}_0(t_0, t)$ by the equation

$$\nabla_0 \mathbf{A}_{[M]} = \mathbf{V}_0 \times \mathbf{A}_{[M]}. \tag{1.9}$$

From this result we deduce the equation

$$\mathbf{V}_0 = \frac{1}{2} \mathbf{A}^{[M]} \times \nabla_0 \mathbf{A}_{[M]}, \tag{1.10}$$

which enables us, like the analogous Eq. (1.4), to express the angular velocity of the basis in terms of the rotation vector (see also [3]):

$$\mathbf{V}_0 = \frac{1}{f} \left(\nabla_0 \mathbf{V} + \frac{1}{2} \mathbf{V} \times \nabla_0 \mathbf{V}\right). \tag{1.11}$$

The explicit analogy between expressions (1.3)-(1.5) and (1.9)-(1.11) provides a means for verifying, first the equivalence of the time differentiation and covariant coordinate differentiation operations (and, hence, the commutativity of the operators $\nabla_0$ and $\nabla_{(L)}$ ) and, second, the mechanical significance of the curvature-strain tensor as the tensor of angular velocities of rigid rotation of the basis, corresponding to infinitesimal rotations of the coordinates at a fixed time.

Time differentiation of Eq. (1.4) with the use of (1.9) yields the equation

$$\nabla_0 \mathbf{V}_{(L)} = \nabla_L \mathbf{V}_0 + \mathbf{V}_0 \times \mathbf{V}_{(L)}, \tag{1.12}$$

which describes the time rates of change of the curvature-strain vectors.

The time rates of change of the metric-strain vectors are similarly determined from Eq. (1.6):

$$\nabla_0 \mathbf{U}_{(L)} = \nabla_L \mathbf{U}_0 - \mathbf{V}_0 \times \mathbf{A}_{[L]}. \tag{1.13}$$

The vector $\mathbf{U}_0(t_0, t)$, defined by the relation

$$\mathbf{U}_0 = \nabla_0 \mathbf{U},$$

has the significance of the linear velocity vector of the basis.

2. To derive the dynamical equations for a thermomechanical couple-stress continuum we define on the instantaneous coordinate surfaces the vectors $\mathbf{X}^{(M)}(t_0, t)$ of force-stresses and vectors $\mathbf{Y}^{(M)}(t_0, t)$ of couple-stresses, all computed per unit initial area.

In addition, let $\mathbf{F}(t_0, t), \mathbf{G}(t_0, t)$ be the vectors of external body forces and couples per unit initial volume, $H_0(t_0, t)$ the rate of heat input to unit initial volume, $\rho(t)$ the initial density of the continuum, and $\mathbf{R}(t)$ the tensor of local moments of inertia (which by definition does not depend on the time and is a measure, averaged over initial unit volume, of the inertia of structural particles of the medium when they execute rotational motion as rigid bodies about the local axes).

At the initial time we isolate in the continuum an arbitrary spatial domain A with a smooth surface B, on which we define the field of unit normal vectors

$$\mathbf{N}(t) = N_{(M)}(t)\mathbf{A}^{(M)}.$$

At any instant the given domain is acted upon by surface and body forces. The principal vectors of surface forces and couples acting externally on the given domain through a surface element $d\beta$ are equal to, respectively, $\mathbf{X}^{(M)}N_{(M)}\,d\beta$ and $\mathbf{Y}^{(M)}N_{(M)}\,d\beta$. The principal vectors of body forces and couples acting on a volume element $d\alpha$ are made up of the vectors $\mathbf{F}d\alpha$, $\mathbf{G}d\alpha$ of external forces and couples and the vectors $\rho\nabla_0\mathbf{U}_0 d\alpha$, $\mathbf{R}\cdot\nabla_0\mathbf{V}_0 d\alpha$ of inertial forces and couples.

The motion of the given domain of the continuum is described by the following dynamic integral equations for the variation of the linear and angular momenta (relative to the reference point in space):

$$\int_B \mathbf{X}^{(M)}N_{(M)}d\beta + \int_A \mathbf{F}d\alpha = \int_A \rho\nabla_0\mathbf{U}_0 d\alpha,$$

$$\int_B \left(\mathbf{a}_0 \times \mathbf{X}^{(M)} + \mathbf{Y}^{(M)}\right) N_{(M)}d\beta + \int_A (\mathbf{a}_0 \times \mathbf{F} + \mathbf{G})\, d\alpha = \int_A (\rho\mathbf{a}_0 \times \nabla_0\mathbf{U}_0 + \mathbf{R}\cdot\nabla_0\mathbf{V}_0)\, d\alpha.$$

For sufficient smoothness of the given functions the transformations of the surface integrals according to the Gauss–Ostrogradskii formula yields the local equations of motion of the couple-stress continuum:

$$\nabla_{(M)}\mathbf{X}^{(M)} + \mathbf{F} = \rho\nabla_0\mathbf{U}_0, \quad \nabla_{(M)}\mathbf{Y}^{(M)} + \mathbf{A}_{(M)} \times \mathbf{X}^{(M)} + \mathbf{G} = \mathbf{R}\cdot\nabla_0\mathbf{V}_0. \tag{2.1}$$

If $U(t_0, t)$ is the internal energy density of the continuum per unit initial volume, then for the given domain the first law of thermodynamics must hold, formulated by the integral equation

$$\nabla_0 \int_A \left(\frac{1}{2}\rho\mathbf{U}_0\cdot\mathbf{U}_0 + \frac{1}{2}\mathbf{V}_0\cdot\mathbf{R}\cdot\mathbf{V}_0 + U\right) d\alpha = \int_A (\mathbf{F}\cdot\mathbf{U}_0 + \mathbf{G}\cdot\mathbf{V}_0 + H_0)\, d\alpha + \int_B \left(\mathbf{X}^{(M)}\cdot\mathbf{U}_0 + \mathbf{Y}^{(M)}\cdot\mathbf{V}_0\right) N_{(M)}\, d\beta. \tag{2.2}$$

From this result, as a result of transformation of the surface integral according to the Gauss–Ostrogradskii formula and application of Eqs. (2.1), we obtain the local energy equation

$$\nabla_0 U = \mathbf{X}^{(M)}\cdot(\nabla_M\mathbf{U}_0 - \mathbf{V}_0 \times \mathbf{A}_{(M)}) + \mathbf{Y}^{(M)}\cdot\nabla_M\mathbf{V}_0 + H_0. \tag{2.3}$$

If the domain occupied by the continuum is bounded and either the displacement and rotation vectors or the vectors $\mathbf{P}$, $\mathbf{Q}$ of external surface forces and couples are specified at its boundary, then the necessary and sufficient condition for satisfaction of the integral equation (2.2) in the entire domain subject to the additional local constraint (2.3) is satisfaction of Eqs. (2.1) in the interior of the domain and of the following equations (boundary conditions) on the boundary surface:

$$(\mathbf{X}^{(M)}N_{(M)} - \mathbf{P})\cdot\mathbf{U}_0 = 0, \ (\mathbf{Y}^{(M)}N_{(M)} - \mathbf{Q})\cdot\mathbf{V}_0 = 0. \tag{2.4}$$

3. For the scalar representation of the formulated kinematic and dynamic equations we can expand the kinematic and force vectors with respect to both the initial and the rotated bases:

$$\mathbf{U} = U_{(N)}\mathbf{A}^{(N)} = U_{[N]}\mathbf{A}^{[N]}, \ \mathbf{V} = V_{(N)}\mathbf{A}^{(N)} = V_{[N]}\mathbf{A}^{[N]} \ (V_{[N]} \equiv V_{(N)}), \tag{3.1}$$

$$\mathbf{U}_{(M)} = U_{(MN)}\mathbf{A}^{(N)} = U_{(MN]}\mathbf{A}^{[N]}, \ \mathbf{V}_{(M)} = V_{(MN)}\mathbf{A}^{(N)} = V_{(MN]}\mathbf{A}^{[N]},$$

$$\mathbf{X}^{(M)} = X^{(MN)}\mathbf{A}_{(N)} = X^{(MN]}\mathbf{A}_{[N]}, \ \mathbf{Y}^{(M)} = Y^{(MN)}\mathbf{A}_{(N)} = Y^{(MN]}\mathbf{A}_{[N]}.$$

Here the transition from either of these bases to the other is realized by means of Eqs. (1.1) written in the form

$$\mathbf{A}_{[M]} = (A_{(MN)} + W_{(MN)})\,\mathbf{A}^{(N)},\ \mathbf{A}_{(M)} = (A_{(NM)} + W_{(NM)})\mathbf{A}^{[N]}, \tag{3.2}$$

$$W_{(MN)} = \frac{1}{f} D_{(LMN)} V^{(L)} + \frac{1}{2f}\left(V_{(M)}V_{(N)} - A_{(MN)}V_{(L)}V^{(L)}\right).$$

Equations (1.12) and (1.13) enable us to establish the relations

$$\nabla_M \mathbf{V}_0 = \nabla_0 V_{(MN]}\,\mathbf{A}^{[N]},\ \nabla_M \mathbf{U}_0 - \mathbf{V}_0 \times \mathbf{A}_{(M)} = \nabla_0 U_{(MN]}\mathbf{A}^{[N]}, \tag{3.3}$$

which can be used to transform Eq. (2.3) to the form

$$\nabla_0 U = X^{(MN]}\,\nabla_0 U_{(MN]} + Y^{(MN]}\nabla_0 V_{(MN]} + H_0, \tag{3.4}$$

showing that the components of the force and strain tensors in the rotated basis are, respectively, generalized internal forces and generalized displacements. The components of these tensors in the initial basis are not so identified.

To close the formulated system of kinematic and dynamic equations it is necessary to invoke the constitutive equations for the couple-stress continuum. The problem of formulating these equations cannot be solved in a general setting. It requires particularization on the basis of experimental work. In particular, for the couple-stress continuum, as for the couple-free continuum, we can identify a class of invertible deformation processes [4]. If we confine the discussion strictly to such processes, then for the formulation of the constitutive equations it is sufficient to specify either the internal energy density U as a function of both strain tensors and the entropy density S:

$$U = U(U_{(MN]},\ V_{(MN]},\ S),$$

or the free-energy density V as a function of the same tensors and the absolute temperature T:

$$V \equiv U - TS = V(U_{(MN]},\ V_{(MN]},\ T).$$

In this case the state parameters S and T are interrelated by the second law of thermodynamics:

$$H_0 = T\nabla_0 S. \tag{3.5}$$

As a result of time differentiation of each of the functions U, V and comparison with Eqs. (3.4) and (3.5), we obtain the differential equations

$$\frac{\partial U}{\partial U_{(MN]}}\nabla_0 U_{(MN]} + \frac{\partial U}{\partial V_{(MN]}}\nabla_0 V_{(MN]} + \frac{\partial U}{\partial S}\nabla_0 S = X^{(MN]}\nabla_0 U_{(MN]} + Y^{(MN]}\nabla_0 V_{(MN]} + T\nabla_0 S,$$

$$\frac{\partial V}{\partial U_{(MN]}}\nabla_0 U_{(MN]} + \frac{\partial V}{\partial V_{(MN]}}\nabla_0 V_{(MN]} + \frac{\partial V}{\partial T}\nabla_0 T = X^{(MN]}\nabla_0 U_{(MN]} + Y^{(MN]}\nabla_0 V_{(MN]} - S\nabla_0 T,$$

from which we deduce equivalent formulations of the constitutive equations and heat-input equations:

$$X^{(MN]} = \frac{\partial U}{\partial U_{(MN]}},\quad Y^{(MN]} = \frac{\partial U}{\partial V_{(MN]}},\quad H_0 = \frac{\partial U}{\partial S}\nabla_0 S; \tag{3.6}$$

$$X^{(MN]} = \frac{\partial V}{\partial U_{(MN]}},\quad Y^{(MN]} = \frac{\partial V}{\partial V_{(MN]}},\quad H_0 = -T\nabla_0 \frac{V}{\partial T}. \tag{3.7}$$

The first formulation is preferable for adiabatic, and the second for isothermal deformation processes.

## 4. The kinematic constraints

$$\mathbf{U}_{(M)}\cdot\mathbf{A}_{[N]} \equiv \mathbf{U}_{(N)}\cdot\mathbf{A}_{[M]}, \tag{4.1}$$

which guarantee symmetry of the metric-strain tensors, yield the Cosserat continuum ("pseudocontinuum") model [1, 4, 5]. This continuum has the same kinematics as the classical continuum, since Eqs. (4.1) describe the rotation vector of a material point of the continuum in terms of its displacement vector in exactly the same way as for the classical model. Only the presence of couple-stresses distinguishes the Cosserat pseudocontinuum from the classical model. The number of independent boundary conditions is reduced to five in the pseudocontinuum model [5].

Augmenting the constraints (4.1) with the dynamic constraints

$$\mathbf{Y}^{(M)}\cdot\mathbf{A}^{[N]} \equiv 0\ ,$$

we obtain a nonlinear model of a couple-free continuum in a formulation unrestricted by the condition of smallness of the local rotations and thereby generalizing the formulation of Biot [2].

Thus, the proposed technique for the construction of nonlinear models of deformable media provides a unified kinematic foundation for couple-stress and couple-free media in an ultimately simple (vector) representation.

## LITERATURE CITED


1. R. A. Toupin, "Theories of elasticity with couple-stresses," Arch. Rat. Mech., Anal., 17, 85-112 (1964).
2. M. A. Biot, "Nonlinear theory of elasticity and the linearized case for a body under initial stress," Philos. Mag., Ser. 7, 27, No. 183 (1939).
3. A. I. Lur'e, Analytical Mechanics [in Russian], Fizmatgiz, Moscow (1961).
4. W. Nowacki, Theory of Elasticity [Russian translation], Mir, Moscow (1975).
5. W. T. Koiter, "Couple stresses in the theory of elasticity," Nederl. Akad. Wetensch. Proc., Ser. B., 67, 17-29; 30-34 (1964).


# ASYMPTOTIC BEHAVIOR OF BOUNDARY-VALUE PROBLEMS FOR AN ELASTIC RING REINFORCED WITH VERY RIGID FIBERS


**Yu. A. Bogan**


**UDC 539.3**

The following boundary-value problems are investigated for an elastic ring reinforced with very rigid fibers arranged in concentric circles: a) The stresses are given at the boundary; b) the bending deflection and angle of rotation are given at the boundary. The generalized Hooke's law [1] is adopted as the initial governing equations; as a consequence, the final results are valid for standard models of a composition elastic model [2, 3].

We formulate asymptotic representations of the solutions of boundary-value problems a) and b) on the assumption that the rigidity of the material in the circumferential direction is much greater than the shear rigidity.

We show that a boundary layer sets in along the boundary; in case a) the boundary conditions for the limiting boundary-value problem do not coincide with any of the boundary conditions for the sublimiting problem. Problem b) degenerates into the limiting problem in a regular manner.

1. Let us consider problem a). We assume that the elastic ring is cylindrically orthotropic, and we apply the generalized Hooke's law in the form [$(r, \theta)$ denotes polar coordinates]

$$\sigma_r = c_{11}\varepsilon_r + c_{12}\varepsilon_\theta,\ \sigma_\theta = c_{12}\varepsilon_r + c_{22}\varepsilon_\theta,\ \ \tau_{r\theta} = c_{66}\gamma_{r\theta}.$$

We introduce the dimensionless stresses and rigidities, setting

$$\bar{\sigma}_r = \sigma_r c_{66}^{-1},\quad \bar{\sigma}_\theta = \sigma_\theta c_{66}^{-1},\quad \bar{\tau}_{r\theta} = \tau_{r\theta} c_{66}^{-1},\quad d_{ij} = c_{ij} c_{66}^{-1},\quad i, j = 1, 2,$$

and in all that follows we retain the same notation as before for the dimensionless stresses. Let $d_{22} \gg 1$; in real situations this relation holds for an elastic ring reinforced with one very rigid set of fibers r = const. We put $\varepsilon^2 = d_{22}^{-1}$, $d = d_{11}^{-1}$, $c = d_{12}^2 + 2d_{12}$, $t = \ln r$. Then the equation for the stress function $w(t, \theta)$ can be written in the form

$$\varepsilon^2 N(w) + M(w) = 0 \tag{1.1}$$

on the assumption that mass forces are absent. In (1.1)

$$N(w) = \frac{\partial^4 w}{\partial t^4} - 4\frac{\partial^3 w}{\partial t^3} + 5\frac{\partial^2 w}{\partial t^2} + 2\frac{\partial w}{\partial t} + 2dc\frac{\partial^3 w}{\partial t\partial\theta^2} - dc\frac{\partial^4 w}{\partial t^2\partial\theta^2} - dc\frac{\partial^2 w}{\partial\theta^2},$$

$$M(w) = d\frac{\partial^4 w}{\partial\theta^4} + \frac{\partial^4 w}{\partial t^2\partial\theta^2} - 2\frac{\partial^3 w}{\partial t\partial\theta^2} - d\frac{\partial^2 w}{\partial t^2} + 2d\frac{\partial w}{\partial t} + (1 + 2d)\frac{\partial^2 w}{\partial\theta^2}.$$


Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 118-122, November-December, 1980. Original article submitted October 29, 1979.